ISRAEL JOURNAL OF MATHEMATICS **167** (2008), 141–154
DOI: 10.1007/s11856-008-1044-8

WEBB'S CONJECTURE FOR FUSION SYSTEMS

BY

Assaf Libman*

Department of Mathematical Sciences, King's College, University of Aberdeen
Aberdeen AB24 3UE , Scotland, U.K.
e-mail: a.libman@abdn.ac.uk

ABSTRACT

We give a short and conceptual proof of Webb's conjecture. Our methods are general enough to prove an analogue of the conjecture for saturated fusion systems.

1. Introduction

Let G be a finite group. The set $S_p^0(G)$ of all the non-trivial p-subgroups of G is partially ordered by inclusion. The partial order is preserved by conjugation in G and there results a geometric simplicial complex $|S_p^0(G)|$ on which G acts. This complex was first studied by Brown [4],[3], see also [8].

Webb proved that the orbit space $|S_p^0(G)|/G$ is $\mathbb{F}_p$-acyclic and conjectured that it is contractible. This fact was proven by Symonds in [11]. His proof hinges on Whitehead's theorem which is applied to a certain homotopy equivalent subspace of $|S_p(G)|$ which is shown to be simply connected and acyclic. In this note we will obtain a conceptual proof of Symonds' theorem.

1.1. Theorem (Symonds [11]): *Let G be a finite group and let $\mathcal{C}$ be a non-empty collection of p-subgroups which is closed under taking super-p-groups. Then $|\mathcal{C}|/G$ is contractible.*

* The author was supported by an EPSRC grant EP/D506484/1.
Received November 24, 2006 and in revised form February 6, 2007

Proof. This is immediate from Theorem 3.2 and the remarks in the beginning of §3, taking into account that geometric realisation commutes with formation of orbit spaces. ∎

A Sylow p-subgroup P of a finite group G gives rise to a small category $\mathcal{F}_P(G)$ whose objects are the p-subgroups of G which are contained in P and whose morphisms are the homomorphisms between them which are restrictions of the inner automorphisms of G. The orbit space $|\mathcal{C}|/G$ for any collection $\mathcal{C}$ of p-subgroups can be easily constructed from $\mathcal{F}_P(G)$ using Sylow's theorems. Abstracting the construction of $\mathcal{F}_P(G)$ leads to the concept of a fusion system on P, which consists of the subgroups of P and group monomorphisms between them, not necessarily induced by a finite group G. A fusion system $\mathcal{F}$ on P is called saturated if it satisfies a certain set of axioms which makes it "look like" a fusion systems associated to a finite group. A full exposition is given in §3 of this paper.

A considerable amount of effort was needed by Linckelmann in [7] to use Symonds's ideas to prove a variant of Webb's conjecture for saturated fusion systems. Proposition 2.13, which is the main observation of this note, allows us to reprove his theorem in an elegant and conceptual way. Specifically,

1.2. *Definition:* Fix a fusion system $\mathcal{F}$ on P. An **$\mathcal{F}$-collection** is a union of $\mathcal{F}$-conjugacy classes of subgroups of P. An $\mathcal{F}$-collection $\mathcal{C}$ is **closed** if a subgroup $Q \leq P$ belongs to $\mathcal{C}$ whenever it contains an element of $\mathcal{C}$.

Clearly, an $\mathcal{F}$-collection is partially ordered by inclusion and gives rise to an ordered simplicial complex $\mathcal{C}$. Its k-simplices are the chains $P_0 < \cdots < P_k$ of proper inclusions of elements of $\mathcal{C}$. There is an equivalence relation on $\mathcal{C}$ where $P_0 < \cdots < P_k$ is equivalent to $P'_0 < \cdots < P'_k$ if there exists a morphism $\varphi : P_k \to P'_k$ in $\mathcal{F}$ such that $\varphi(P_i) = P'_i$. This yields a quotient space $|\mathcal{C}|/\mathcal{F}$. We shall prove in §3

1.3. Theorem (Linckelmann [7]): *Let $\mathcal{F}$ be a saturated fusion system on P and let $\mathcal{C}$ be a non-empty closed $\mathcal{F}$-collection. Then $|\mathcal{C}|/\mathcal{F}$ is contractible.*

We shall also prove

1.4. Theorem: *Fix a saturated fusion system $\mathcal{F}$ on P and let $\mathcal{E}$ denote the $\mathcal{F}$-collection of the non-trivial elementary abelian p-subgroups of P. Then $|\mathcal{E}|/\mathcal{F}$ is contractible.*

2. Sylow p-subgroups and orbit spaces

Throughout this section G is a discrete group, not necessarily finite. We shall adopt the convention that a space is a simplicial set, see e.g. [1, Chapter VIII] or [6, Theorems I.11.3 and 11.4]. A G-space is therefore a simplicial G-set.

Recall that $\Delta[0]$ and $\Delta[1]$ denote the standard simplicial 0-simplex and 1-simplex. There are standard inclusions of "endpoints" $d^0, d^1 : \Delta[0] \to \Delta[1]$ (cf. [1, pp. 234–235]).

2.1. *Definition:* Consider two G-maps $f, g : X \to Y$. We say that f is **homotopic to** g, written $f \sim g$, if there exists a G-map (a homotopy) $h : X \times \Delta[1] \to Y$ such that $f = (1_X \times d^1) \circ h$ and $g = (1_X \times d^0) \circ h$ (cf. [1, p. 245]). We say that f and g are **weakly G-homotopic** if they become equivalent under the equivalence relation generated by $\sim$ on the set of G-maps $X \to Y$ (cf. [6, I.6 or II.1.5]). A map $f : X \to Y$ is a **weak G-homotopy equivalence** if there exists $g : Y \to X$ such that both $f \circ g$ and $g \circ f$ are weakly G-homotopic to the respective identities.

Clearly, weakly G-homotopic maps $f, g : X \to Y$ induce (weakly) homotopic maps $f : X/G \to Y/G$ on orbit spaces. Consequently, a weak G-homotopy equivalence $f : X \to Y$ induces a weak homotopy equivalence $f : X/G \to Y/G$.

2.2. *Definition:* $\mathrm{Iso}_G(X)$ is the set of isotropy groups of the simplices of a G-space X.

Clearly, if H belongs to $\mathrm{Iso}_G(X)$, then so do all its conjugates.

2.3. *Definition:* A **collection** in G is a set $\mathcal{H}$ which is a union of conjugacy classes of subgroups of G. A collection $\mathcal{H}$ is **finite** if it consists of finitely many conjugacy classes. We say that the **orbit type** of a G-space X is in $\mathcal{H}$ if $\mathrm{Iso}_G(X) \subseteq \mathcal{H}$. If $\mathcal{H}$ is finite, we say that X is of **finite orbit type**.

2.4. *Definition:* A subcollection $\mathcal{H}'$ of $\mathcal{H}$ is **closed** if no element of $\mathcal{H} \backslash \mathcal{H}'$ contain an element of $\mathcal{H}'$.

Let G/H denote the set of right cosets of $H \leq G$ with the obvious G-action by left translation. Note that for $K \leq G$ the fixed points set $(G/H)^K$ is

$$(G/H)^K = \{gH \ : \ g^{-1}Kg \leq H\}.$$

It follows immediately that

2.5. PROPOSITION: *A subcollection $\mathcal{H}'$ of $\mathcal{H}$ is closed if and only if $(G/H)^K$ is empty whenever $H \in \mathcal{H} \backslash \mathcal{H}'$ and $K \in \mathcal{H}'$. Equivalently,* $\mathrm{Hom}_G(G/K, G/H)$ *is empty.*

In light of this observation we see that the face and degeneracy maps in a G-space X (namely, a simplicial G-set) must carry n-simplices whose isotropy group is in $\mathcal{H}'$ to k-simplices with the same property. The following definition now makes sense:

2.6. *Definition:* Let $\mathcal{H}'$ be a closed subcollection of $\mathcal{H}$ and let X be a G-space of orbit type $\mathcal{H}$. Let $X_{\mathcal{H}'}$ denote the G-subspace of X consisting of the simplices whose isotropy group belongs to the subcollection $\mathcal{H}'$ of $\mathcal{H}$.

The same observation Proposition 2.5 and the definition of weak G-homotopy Definition 2.1 makes the following proposition clear. It should be compared with [13, §I.6].

2.7. PROPOSITION: *Let $\mathcal{H}'$ be a closed subcollection of $\mathcal{H}$ in G. Then the assignment $X \mapsto X_{\mathcal{H}'}$ defines a functor*

$$\{G\text{-spaces of orbit type } \mathcal{H}\} \to \{G\text{-spaces of orbit type } \mathcal{H}'\}.$$

It preserves weakly G-homotopic maps and, consequently, weak G-homotopy equivalences. The inclusion $X_{\mathcal{H}'} \subseteq X$ provides a natural transformation to the identity functor.

Here is a useful example of closed subcollections.

2.8. PROPOSITION: *Let $\mathcal{H}$ be a collection in G. Fix a prime p and an integer n and let $\mathcal{H}'$ denote the subcollection of $\mathcal{H}$ consisting of the subgroup $H \in \mathcal{H}$ which contain a finite p-subgroup of order $> p^n$. Then $\mathcal{H}'$ is closed in $\mathcal{H}$.*

Proof. By definition if $H \in \mathcal{H} \backslash \mathcal{H}'$ then the order of every finite p-subgroup of H must be $\leq p^n$ hence it cannot contain a group $K \in \mathcal{H}'$. ∎

2.9. *Definition:* A finite p-subgroup P of G is a **Sylow p-subgroup** of G if every finite p-subgroup of G is conjugate to a subgroup of P.

Clearly any two Sylow p-subgroups of G are conjugate. In general, if G contains a Sylow p-subgroup, this need not be the case for the subgroups of G.

2.10. *Definition:* Let $\mathcal{P}$ be a collection of finite p-subgroups of G. We say that a collection $\mathcal{H}$ in G has **p-type** $\mathcal{P}$ if every $H \in \mathcal{H}$ contains a Sylow p-subgroup which belongs to $\mathcal{P}$. We write this condition $\mathrm{Syl}_p(\mathcal{H}) \subseteq \mathcal{P}$. A G-space X has p-type $\mathcal{P}$ if the collection $\mathrm{Iso}_G(X)$ has p-type $\mathcal{P}$.

A collection $\mathcal{P}$ of finite p-subgroups of G is partially ordered by inclusion and clearly contains minimal elements provided $\mathcal{P}$ is not empty.

2.11. Proposition: *Let $\mathcal{H}$ be a collection in G of p-type $\mathcal{P}$ where $\mathcal{P}$ is not empty. Let Q be a minimal element in $\mathcal{P}$ and let $\mathcal{P}'$ denote the collection $\mathcal{P}\backslash\{(Q)\}$, that is $\mathcal{P}'$ is obtained from $\mathcal{P}$ by removing the conjugacy class of Q. Then*

$$\mathcal{H}' := \{H \in \mathcal{H} \ : \ Q \text{ is not conjugate to a Sylow } p\text{-subgroup of } H\}.$$

is a closed subcollection of $\mathcal{H}$ of p-type $\mathcal{P}'$. Moreover, for every $R \in \mathcal{P}'$ and every $H \in \mathcal{H}\backslash\mathcal{H}'$ the set $(G/H)^R$ is empty.

Proof. Choose $K \in \mathcal{H}'$ and let R be a Sylow p-subgroup of K. Note that $R \in \mathcal{P}'$ because R is not conjugate to Q by definition of $\mathcal{H}'$. This shows that $\mathrm{Syl}_p(\mathrm{Iso}_G(\mathcal{H}')) \subseteq \mathcal{P}'$.

To show that $\mathcal{H}'$ is closed in $\mathcal{H}$ we choose $H \in \mathcal{H}\backslash\mathcal{H}'$ and prove that (see Proposition 2.5) $(G/H)^K$ is empty. It suffices to show that $(G/H)^R$ is empty, which we proceed to do.

If $(G/H)^R$ is not empty then R is conjugate to a subgroup of H which may be assumed to contain Q as a Sylow p-subgroup. Hence, R is conjugate to a subgroup of Q and therefore R must be conjugate to Q by the minimality of Q in $\mathcal{P}$. This is a contradiction since $R \in \mathcal{P}'$. ∎

Recall that X^H is naturally an N_GH-space for every G-space X and every $H \leq G$. Thus, $(G/K)^H$ is, in general, a union (possibly infinite) of N_GH-orbits. The following is a simple, and essentially well-known, observation.

2.12. Lemma: *Let H be a subgroup of G and assume that H contains a Sylow p-subgroup P. Then $(G/H)^P$ is isomorphic to the N_GP-orbit N_GP/N_HP.*

Proof. If $gH \in (G/H)^P$, then $P^g \leq H$ and therefore $P^g = P^h$ for some $h \in H$. It follows that $g \in N_GP \cdot H$, consequently $(G/H)^P \subseteq N_GP \cdot H/H$. The opposite inclusion is obvious. ∎

Recall that the fibred product $X \times_G Y$ of G-spaces X and Y is the orbit space of $X \times Y$ under the diagonal action of G. In particular, when X is a K-space where $K \leq G$ there results a G-space $G \times_K X$. There is a natural isomorphism $(G \times_K X)/G \cong X/K$. When X is a G-space, there is a natural map $G \times_{N_G K} (X^K) \to X$. It is an isomorphism for $X = G/K$.

The main result of this section is Proposition 2.13 below. It allows to determine when a G-map $X \to Y$ induces a weak homotopy equivalence $X/G \to Y/G$ by only checking fixed point subspaces under p-subgroups.

2.13. Proposition: *Let $f : X \to Y$ be a map of G-spaces of finite p-type $\mathcal{P}$. Assume that for every $Q \in \mathcal{P}$ the induced map $f^Q : X^Q \to Y^Q$ is a weak $N_G Q$-homotopy equivalence Definition 2.1. Then f induces a weak homotopy equivalence $X/G \to Y/G$.*

Proof. We prove the result by induction on $|\mathcal{P}|$. If $\mathcal{P}$ is empty then X and Y are empty and the result is trivial.

Fix a non-empty $\mathcal{P}$ and assume by induction that the result holds for all maps f between G-spaces of p-type $\mathcal{P}'$ where $\mathcal{P}'$ is properly contained in $\mathcal{P}$. Consider $f : X \to Y$ between G-spaces of p-type $\mathcal{P}$ and let $\mathcal{H}$ denote $\mathrm{Iso}_G(X \sqcup Y)$. Let Q be a minimal element in $\mathcal{P}$ and consider the collections $\mathcal{P}' = \mathcal{P} \backslash \{(Q)\}$ and $\mathcal{H}'$ as in Proposition 2.11. Set $X' = X_{\mathcal{H}'}$ and $Y' = Y_{\mathcal{H}'}$. Proposition 2.7 shows that f gives rise to the following morphism of commutative squares.

(1)
$$\begin{array}{ccc} G \times_{NQ} (X'^Q) & \longrightarrow & X' \\ \Big\downarrow & & \Big\downarrow \\ G \times_{NQ} (X^Q) & \longrightarrow & X \end{array} \quad \xrightarrow{f} \quad \begin{array}{ccc} G \times_{NQ} (Y'^Q) & \longrightarrow & Y' \\ \Big\downarrow & & \Big\downarrow \\ G \times_{NQ} (Y^Q) & \longrightarrow & Y. \end{array}$$

Observe that $\mathrm{Iso}_G(X') \cup \mathrm{Iso}_G(Y') \subseteq \mathcal{H}'$ whose p-type is $\mathcal{P}'$. Note that if $R \in \mathcal{P}'$ then ${X'}^R = X^R$ because for every $H \in \mathrm{Iso}_G(X) \backslash \mathrm{Iso}_G(X') = \mathcal{H} \backslash \mathcal{H}'$ we have $(G/H)^R = \emptyset$, by Proposition 2.11. Similarly ${Y'}^R = Y^R$ and since $X^R \to Y^R$ is a weak $N_G R$-homotopy equivalence by hypothesis, we see that ${X'}^R \to {Y'}^R$ is a weak $N_G R$-homotopy equivalence for all $R \in \mathcal{P}'$. The induction hypothesis applies to $f' : X' \to Y'$ and $\mathcal{P}'$ and therefore

(2) $$X'/G \to Y'/G \quad \text{is a weak homotopy equivalence.}$$

Consider the $N_G Q$-spaces X^Q and Y^Q and set

$$\mathcal{K} = \mathrm{Iso}_{NQ}(X^Q \sqcup Y^Q).$$

By hypothesis $X^Q \to Y^Q$ is a weak NQ-homotopy equivalence and in particular

$$X^Q/NQ \to Y^Q/NQ \quad \text{is a weak homotopy equivalence.} \tag{3}$$

Consider

$$\mathcal{K}' := \{K \in \mathcal{K} : K \text{ contains a finite } p\text{-group whose order is } > |Q|\}.$$

This is a closed subcollection of $\mathcal{K}$ by proposition 2.8. Proposition 2.7 shows that

$$(X^Q)_{\mathcal{K}'} \to (Y^Q)_{\mathcal{K}'} \quad \text{is a weak } NQ\text{-homotopy equivalence.} \tag{4}$$

We now claim that $(X^Q)_{\mathcal{K}'} = (X')^Q$. Choose a simplex $\sigma \in (X')^Q$ and set $H = G_\sigma$. Observe that $Q \leq H$ and that $H \in \mathcal{H}'$. Therefore $Q \lneq S$, where $S \in \mathcal{P}'$ is a Sylow p-subgroup of H. It follows that $Q \lneq N_S(Q) \leq N_H(Q)$, whence $G_\sigma \cap N_G Q = N_H(Q) \in \mathcal{K}'$. We see that $\sigma \in (X^Q)_{\mathcal{K}'}$ and we have therefore proved that $(X')^Q \subseteq (X^Q)_{\mathcal{K}'}$. Conversely, choose a simplex $\sigma \in (X^Q)_{\mathcal{K}'}$ and set $H = NQ$. Then Q is a proper subgroup of a finite p-subgroup Q' of the isotropy group $H_\sigma = H \cap G_\sigma$. In particular $Q \leq G_\sigma$ is not a Sylow p-subgroup of G_σ, so that $\sigma \in (X')^Q$. We deduce that $(X^Q)_{\mathcal{K}'} \subseteq (X')^Q$ and therefore equality holds.

Similarly $(Y^Q)_{\mathcal{K}'} = (Y')^Q$ and together with (4) we see that $(X')^Q \to (Y')^Q$ is a weak NQ-homotopy equivalence, so in particular

$$(X')^Q/NQ \to (Y')^Q/NQ \quad \text{is a weak homotopy equivalence.} \tag{5}$$

By taking G-orbits in (1) we obtain the following morphism of commutative squares.

$$\begin{tikzcd} X'^Q/NQ \arrow[r] \arrow[d, hook] & X'/G \arrow[d, hook] \\ X^Q/NQ \arrow[r] & X/G \end{tikzcd} \xrightarrow{f} \begin{tikzcd} Y'^Q/NQ \arrow[r] \arrow[d, hook] & Y'/G \arrow[d, hook] \\ Y^Q/NQ \arrow[r] & Y/G. \end{tikzcd} \tag{6}$$

The vertical arrows are clearly inclusion of spaces because X' (resp. Y') is a G-subspace of X (resp. Y) and similarly $(X')^Q = (X^Q)_{\mathcal{K}'}$ is an NQ-subspace of X^Q (resp. $(Y')^Q$ is an NQ-subspace of Y^Q). Thus, the vertical arrows in (6) are cofibrations (see, e.g., [1, p. 240], or [6, Theorem I.11.3])

Now, in every simplicial dimension n, a G-orbit in $X_n \backslash X'_n$ is isomorphic to G/H where $H \in \mathcal{H} \backslash \mathcal{H}'$ and, moreover, H may be assumed to contain Q as a

Sylow p-subgroup. Lemma 2.12 implies that $G \times_{NQ} (G/H)^Q \cong G/N_H Q$, hence

$$(X_n \backslash X'_n)^Q/NQ \to (X_n \backslash X'_n)/G \quad \text{and} \quad (Y_n \backslash Y'_n)^Q/NQ \to (Y_n \backslash Y'_n)/G$$

are bijections. Therefore, (6) is a morphism of pushout squares because this is the case in every simplicial degree. It follows from Corollary II.8.6 and Lemma II.8.12 in [6] that $X/G \to Y/G$ is a weak homotopy equivalence because the arrows (2), (5) and (3) are weak homotopy equivalences. Alternatively, one may argue that (6) is a morphism of homotopy-pushout squares because the vertical arrows are cofibrations. This completes the induction step. ∎

3. Webb's conjecture and fusion systems

Let $\mathcal{X}$ be a poset with an action of a group G. It gives rise to a small category where $x_0 \leq x_1$ corresponds to a morphism $x_0 \to x_1$. One obtains a simplicial G-set $\mathrm{Nr}\,\mathcal{X}$, called the nerve of $\mathcal{X}$, whose set of n-simplices is the set of all the n composable morphisms $x_0 \to x_1 \to \cdots \to x_n$ (see [1, p. 291]). The geometric realisation of $\mathrm{Nr}\,\mathcal{X}$ is G-equivariantly homeomorphic to the geometric G-simplicial complex associated to the G-poset $\mathcal{X}$ whose set of n-simplices is indexed by the set $\{x_0 \lneq x_1 \lneq \cdots \lneq x_n\}$. Since in this note we model topological spaces by simplicial sets, we shall write $|\mathcal{X}|$ for the nerve of $\mathcal{X}$ (rather than its geometric realisation).

3.1. *Definition:* Let $S_p(G)$ denote the collection of all the finite p-subgroups of G and $S_p^0(G)$ the collection of the non-trivial finite p-subgroups.

Recall from Definition 2.4 that a closed subcollection $\mathcal{C}$ of $S_p(G)$ is a collection of finite p-groups of G such that if $Q \in \mathcal{C}$ and Q' is a finite p-subgroup of G containing Q then $Q' \in \mathcal{C}$.

3.2. Theorem: *Let G be a group which contains a Sylow p-subgroup P and let $\mathcal{C}$ be a non-empty closed subcollection of $S_p(G)$. Assume that for every $P_0 \leq \cdots \leq P_k$ in $\mathcal{C}$ the group $\bigcap_{i=0}^k N_G(P_i)$ contains a Sylow p-subgroup. Then $|\mathcal{C}|/G$ is contractible.*

Proof. Let X denote $|\mathcal{C}|$. The isotropy group of a k-simplex $P_0 \leq \cdots \leq P_k$ in $|\mathcal{C}|$ is the group $H = \bigcap_{i=0}^k N_G(P_i)$ which by hypothesis contains a Sylow p-subgroup Q. Clearly $P_0 \lhd H$ and therefore $P_0 \leq Q$ which implies, in turn, that $Q \in \mathcal{C}$ because P_0 belongs to $\mathcal{C}$ which is closed in $S_p(G)$. We deduce that X has p-type $\mathcal{C}$.

Let Y denote the one-point G-space. Clearly, its p-type is $\{(P)\}$ which is contained in $\mathcal{C}$ because the latter is not empty and closed in $S_p(G)$.

We shall now consider the canonical map $f : X \to Y$. We have seen that X and Y have p-type $\mathcal{C}$ which must be finite because P is a finite group and it contains an element from every conjugacy class in $S_p(G)$. In order to apply Proposition 2.13 it remains to show that X^Q is weakly NQ-equivalent to a point for every $Q \in \mathcal{C}$.

Note that $\mathcal{C}^Q$ consists of $P' \in \mathcal{C}$ such that $Q \leq N_G P'$. The zigzag of inclusions

$$P' \leq P'Q \geq Q, \quad P' \in \mathcal{C}^Q$$

provides a zigzag of natural transformations which connect the identity on $\mathcal{C}^Q$ to the constant functor on the object Q. Moreover, by inspection these functors and natural transformations are NQ-equivariant because NQ fixes Q. Upon taking nerves, there results a weak NQ-homotopy from the identity of $|\mathcal{C}^Q|$ to the constant self map. In other words, $|\mathcal{C}^Q|$ is weakly NQ-equivalent to a point. Finally, observe that $|\mathcal{C}^Q| = |\mathcal{C}|^Q$, so we can apply Proposition 2.13 and deduce that $f : X \to Y$ induces a weak homotopy equivalence $X/G \to Y/G = *$, that is $|\mathcal{C}|/G$ is contractible. ∎

Fix a finite p-group P. Let $\mathrm{Inj}(Q, Q')$ denote the set of all the injective homomorphisms $Q \to Q'$ for $Q, Q' \leq P$. Let $\mathrm{Hom}_P(Q, Q')$ denote those homomorphisms $Q \to Q'$ that are induced by restriction of inner automorphisms $c_g \in \mathrm{Inn}(P)$ where $c_g : x \mapsto gxg^{-1}$.

A **fusion system** $\mathcal{F}$ on P is a subcategory of the category of groups whose objects are the subgroups of P and such that for every $Q, Q' \leq P$

(a) $\mathrm{Hom}_P(Q, Q') \subseteq \mathcal{F}(Q, Q') \subseteq \mathrm{Inj}(Q, Q')$ and
(b) every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion of groups.

We have used the notation $\mathcal{F}(Q, Q')$ for the morphism set in $\mathcal{F}$ between the objects Q and Q'.

As an example, let G be a group and P a finite p-subgroup of G. There results a fusion system $\mathcal{F} = \mathcal{F}_P(G)$ on P whose morphism sets $\mathcal{F}(Q, Q')$ are the restrictions of the inner automorphisms c_g of G, where $g \in G$, to the subgroups Q and Q' whenever $c_g(Q) \leq Q'$.

3.3. *Definition:* A fusion system $\mathcal{F}$ on P is **realised** by a group G if G contains P as a Sylow p-subgroup (Definition 2.9) and $\mathcal{F} = \mathcal{F}_P(G)$.

We shall now discuss saturated fusion systems. Our treatment follows Broto–Levi–Oliver in [2].

3.4. *Definition:* Fix a fusion system $\mathcal{F}$ on P. Isomorphic objects Q, Q' of $\mathcal{F}$ are called **$\mathcal{F}$-conjugate**. A subgroup $Q \leq P$ is **fully $\mathcal{F}$-centralised** if $|C_P(Q)| \geq |C_P(Q')|$ for any Q' which is $\mathcal{F}$-conjugate to Q. A subgroup $Q \leq P$ is **fully $\mathcal{F}$-normalised** if $|N_P(Q)| \geq |N_P(Q')|$ for any Q' which is $\mathcal{F}$-conjugate to Q.

Given a morphism $\varphi \in \mathcal{F}(Q, P)$ define

$$N_\varphi = \{g \in N_P(Q) : \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q))\}.$$

3.5. *Definition* (cf. [2])*:* A fusion system $\mathcal{F}$ on P is **saturated** if

(I) Any fully $\mathcal{F}$-normalised subgroup $Q \leq P$ is fully $\mathcal{F}$-centralised and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_\mathcal{F}(Q)$.

(II) If $\varphi \in \mathcal{F}(Q, P)$ is such that $\varphi(Q)$ is fully $\mathcal{F}$-centralised then there exists $\bar{\varphi} \in \mathcal{F}(N_\varphi, P)$ which extends φ.

If P is a Sylow p-subgroup of a finite group G then $\mathcal{F}_P(G)$ is saturated. This fails in general if G is not finite. Nevertheless

3.6. Theorem (Robinson [9, Theorem 2]): *Every saturated fusion system is realisable.*

3.7. *Definition:* A group G is called **pseudo finite at** p if for every chain $Q_1 \leq \cdots \leq Q_n$ of inclusions of finite p-subgroups of G, the subgroup $G \cap \bigcap_{i=1}^n N_G Q_i$ contains a Sylow p-subgroup.

Notice that the definition includes the statement that G contains a Sylow p-subgroup by considering the case $n = 0$.

Recall from [2, §6] that a fully $\mathcal{F}$-normalised subgroup P' in a saturated fusion system $\mathcal{F}$, is associated with a normaliser fusion system $N_\mathcal{F}(P')$ on $N_P P'$ whose morphism sets consists of those morphisms $\varphi \in \mathcal{F}(Q, Q')$ which extend to a morphism $\bar{\varphi} : P'Q \to P'Q'$ in $\mathcal{F}$ which satisfies $\bar{\varphi}(P') = P'$. It is shown in [2] that $N_\mathcal{F}(P')$ is saturated if $\mathcal{F}$ is saturated.

3.8. Proposition: *Assume that G realises a saturated fusion system $\mathcal{F}$ on P. Then for every fully $\mathcal{F}$-normalised subgroup $Q \leq P$ the normaliser fusion system $N_{\mathcal{F}}(Q)$ on $N_P Q$ is realised by $N_G Q$.*

Proof. First we show that $N_P Q$ is a Sylow p-subgroup of $N_G Q$. Fix a finite p-subgroup $R \leq N_G Q$. Since Q is fully $\mathcal{F}$-normalised, the image of $N_P Q$ in $N_G Q / C_G Q$ is a Sylow p-subgroup. Therefore, up to conjugation in $N_G Q$ we may assume that

$$R \cdot C_G Q \leq N_P Q \cdot C_G Q. \tag{7}$$

Consider the finite p-group QR. Since P is a Sylow p-subgroup in G there exists some $g \in G$ such that

$$c_g(QR) = g(QR)g^{-1} \leq P.$$

In particular $Q, c_g(Q) \leq P$ so

$$\varphi := c_{g^{-1}} \in \mathrm{Hom}(c_g(Q), Q)$$

is a morphism in $\mathcal{F} = \mathcal{F}_P(G)$. Note that R normalises Q so $c_g(R) \leq N_P(c_g(Q))$. Furthermore, for any $gxg^{-1} \in c_g(R)$ where $x \in R$ we have

$$\varphi \circ c_{gxg^{-1}} \circ \varphi^{-1} = c_{g^{-1}} \circ c_{gxg^{-1}} \circ c_g = c_x \in \mathrm{Aut}_{\mathcal{F}}(Q).$$

Now (7) shows that $x \equiv y \mod CQ$ for some $y \in N_P Q$ and therefore

$$\varphi \circ c_{gxg^{-1}} \circ \varphi^{-1} \in \mathrm{Aut}_P(Q),$$

in other words $c_g(R) \leq N_\varphi$. Since Q is fully $\mathcal{F}$-normalised, it is fully $\mathcal{F}$-centralised and therefore φ extends to a morphism $\tilde{\varphi} : c_g(RQ) \to P$ in $\mathcal{F}$. Since G realises $\mathcal{F}$ there exists some $h \in G$ such that $\tilde{\varphi} = c_h$. Observe that $c_{hg}(Q) = c_h(c_g(Q)) = \tilde{\varphi}(c_g(Q)) = Q$, that is $hg \in N_G Q$. Furthermore $c_{hg}(R) = \tilde{\varphi}(c_g(R)) \leq N_P Q$ because $c_g(R)$ normalises $c_g(Q)$. We have proved, thus, that R is conjugate in $N_G Q$ to a subgroup of $N_P Q$. This shows that $N_P Q$ is a Sylow p-subgroup of $N_G Q$.

It remains to prove that $N_G Q$ realises $N_{\mathcal{F}}(Q)$. First, $N_{\mathcal{F}}(Q)$ and $\mathcal{F}_{N_P Q}(N_G Q)$ are fusion systems on the same group $N_P Q$. Every morphism $c_g : R \to R'$ in $\mathcal{F}_{N_P Q}(N_G Q)$ is a morphism in $\mathcal{F}$ which clearly extends to $c_g : QR \to QR'$. This shows that $\mathcal{F}_{N_P Q}(N_G Q) \subseteq N_{\mathcal{F}}(Q)$. Conversely, a morphism $\varphi : R \to R'$ in $N_{\mathcal{F}}(Q)$ extends, by definition, to a morphism $\psi : QR \to QR'$ in $\mathcal{F}$ such that $\psi(Q) = Q$. Since G realises $\mathcal{F}$ there exists $g \in G$ such that $\psi = c_g$. In particular

$c_g(Q) = Q$, namely $g \in N_G Q$, and by definition $c_g|_R = \psi|_R = \varphi$. This shows that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}_{N_P Q}(N_G Q)$. ∎

3.9. Proposition: *A group which realises a saturated fusion system is pseudo finite at p (Definition 3.7).*

Proof. Let $\mathcal{F}$ be a saturated fusion system on P which is realised by G and consider a chain $Q_1 \leq \cdots \leq Q_n$ of subgroups of P. We have to show that $G \cap (\cap_i N Q_i)$ contains a Sylow p-subgroup.

We prove this by induction on n. The base of induction, $n = 0$, is contained in the definition of realisation of a fusion system. Assume by induction that the result holds for all groups G which realise saturated fusion systems and all chains $Q_1 \leq \cdots \leq Q_{n-1}$ of length $n - 1 \geq 0$ in these groups.

Possibly conjugating the chain $Q_1 \leq \cdots \leq Q_n$ by an element of G, we may assume that Q_n is a fully $\mathcal{F}$-normalised subgroup of P which is a Sylow p-subgroup of G. It is shown in [2, Proposition A.6] that $N_{\mathcal{F}}(Q_n)$ is a saturated fusion system on $N_P Q$ and Proposition 3.8 shows that $N_G Q_n$ realises it. We can apply the induction hypothesis to the chain $Q_1 \leq \cdots \leq Q_{n-1}$ in $N Q_n$ to deduce that

$$\bigcap_{i=1}^{n} N_G Q_i = \bigcap_{i=1}^{n-1} N_{N Q_n}(Q_i)$$

contains a Sylow p-subgroup. ∎

An $\mathcal{F}$-collection (Definition 1.2) is clearly a poset by inclusion of groups. Two k-simplices $P_0 \leq \cdots \leq P_k$ and $P'_0 \leq \cdots \leq P'_k$ in the nerve of $\mathcal{C}$ are **$\mathcal{F}$-conjugate** if there exists $\varphi \in \mathcal{F}(P_k, P'_k)$ such that $\varphi(P_i) = P'_i$ for all $i = 0, \ldots, k$. It is easy to verify that this is an equivalence relation which respects the face and degeneracy maps in $|\mathcal{C}|$. We shall be interested in the quotient space $|\mathcal{C}|/\mathcal{F}$.

Suppose that $\mathcal{F}$ is realised by a group G, that is $\mathcal{F} = \mathcal{F}_P(G)$ where P is a Sylow p-subgroup of G. An $\mathcal{F}$-collection $\mathcal{C}$ gives rise to a collection $\hat{\mathcal{C}}$ in G which consists of the conjugacy classes of the groups in $\mathcal{C}$. There is an obvious inclusion of spaces $|\mathcal{C}| \subseteq |\hat{\mathcal{C}}|$. Observe that any two $\mathcal{F}$-conjugate simplices in $\mathcal{C}$ are G-conjugate because $\mathcal{F} = \mathcal{F}_P(G)$. There results a well-defined map of spaces

$$|\mathcal{C}|/\mathcal{F} \to |\hat{\mathcal{C}}|/G.$$

3.10. Proposition: *Let G realise a fusion system $\mathcal{F}$ on P and let $\mathcal{C}$ be an $\mathcal{F}$-collection. Then $|\mathcal{C}|/\mathcal{F} \to |\hat{\mathcal{C}}|/G$ is an isomorphism (of simplicial sets).*

Proof. Note that P is a Sylow p-subgroup of G so every $P_0 \leq \cdots \leq P_k$ in $|\hat{\mathcal{C}}|$ is G-conjugate to some $P'_0 \leq \cdots \leq P'_k$ where $P'_k \leq P$. Also every P_i is G-conjugate to some $P''_i \in \mathcal{C}$. It follows that P''_i and P'_i are $\mathcal{F}_P(G)$ conjugate so $P'_0 \leq \cdots \leq P'_k$ is a simplex in $|\mathcal{C}|$. This shows that $|\mathcal{C}|/\mathcal{F} \to |\hat{\mathcal{C}}|/G$ is surjective. Any two simplices $P_0 \leq \cdots \leq P_k$ and $P'_0 \leq \cdots \leq P'_k$ in $\mathcal{C}$ which are G-conjugate, are by definition $\mathcal{F}_P(G)$-conjugate. This shows that the map is also injective. ∎

Proof of Theorem 1.3. Theorem 3.6 shows that $\mathcal{F}$ is realised by some G. Also G is pseudo finite at p by Proposition 3.9. Let $\hat{\mathcal{C}}$ be the collection in G consisting of the conjugacy classes of the elements of $\mathcal{C}$. Proposition 3.10 shows that $|\mathcal{C}|/\mathcal{F} \cong |\hat{\mathcal{C}}|/G$. Observe that $\hat{\mathcal{C}}$ is a closed subcollection of $S_p(G)$ because if $Q \in \hat{\mathcal{C}}$ is contained in a p-subgroup $R \leq G$, then $R^g \leq P$ for some $g \in G$. But Q is G-conjugate to some $Q' \in \mathcal{C}$ and therefore Q' is $\mathcal{F}_P(G)$-conjugate to Q^g, namely $Q^g \in \mathcal{C}$. It follows that $R^g \in \mathcal{C}$ because $\mathcal{C}$ is closed, hence $R \in \hat{\mathcal{C}}$.

Also note that $\hat{\mathcal{C}}$ is a finite collection because P is a finite group which contains a representative from every conjugacy class of $S_p(G)$. We can thus apply Theorem 3.2 to deduce that $|\hat{\mathcal{C}}|/G$ is contractible. ∎

Proof of Theorem 1.4. Let $\Omega_1(\Gamma)$ denote the subgroup generated by the elements of order p in a group Γ. Notice that every $H \in \mathrm{Iso}_G(|\hat{\mathcal{E}}|)$ contains a Sylow p-subgroup Q by proposition 3.9. Moreover, Q is not trivial because $E_0 \leq H = \bigcap_i N_G(E_i)$ for some chain $E_0 \leq \cdots \leq E_k$ in $\mathcal{E}$.

To apply proposition 2.13 to $|\hat{\mathcal{E}}| \to *$ it suffices to show that $\mathcal{E}^Q$ is weakly NQ-equivalent to a point for every non-trivial finite p-subgroup Q of G. Indeed there are finitely many conjugacy classes of such Q's because G has a Sylow p-subgroup P. The zigzag of inclusions

$$E \geq C_E(\Omega_1 Z(Q)) \leq C_E(\Omega_1 Z(Q)) \cdot \Omega_1 Z(Q) \geq \Omega_1 Z(Q), \quad E \in \hat{\mathcal{E}}^Q$$

provides a zigzag of natural transformations from the identity on $\hat{\mathcal{E}}^Q$ to a constant endofunctor. Moreover, by inspection this zigzag is NQ-equivariant. This shows that $|\hat{\mathcal{E}}|^Q = |\hat{\mathcal{E}}^Q|$ is weakly NQ-equivalent to a point. This argument is due to Dwyer (in [5, §8]) who attributes it to Quillen.

Alternatively, the result follows by using Theorem 1.3 and the results of [12] which can be used to prove that the inclusion $\hat{\mathcal{E}} \subseteq S_p^0(G)$ induces a weak G-homotopy equivalence on the associated simplicial complexes. ∎

References

[1] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations,* Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[2] C. Broto, R. Levi and B. Oliver, *The homotopy theory of fusion systems,* Journal of the American Mathematical Society **16** (2003), 779–856.

[3] K. S. Brown, *Euler characteristics of discrete groups and G-spaces,* Inventiones Mathematicae **27** (1974), 229–264.

[4] K. S. Brown, *High dimensional cohomology of discrete groups,* Proceedings of the National Academy of Sciences of the United States of America **73** (1976), 1795–1797.

[5] W. G. Dwyer, *Sharp homology decompositions for classifying spaces of finite groups,* in *Group Representations: Cohomology, Group Actions and Topology (Seattle, WA, 1996),* Proceedings of Symposia in Pure Mathematics, vol. 63, American Mathematical Society, Providence, RI, 1998, pp. 197–220,

[6] P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theorym,* Progress in Mathematics, 174, Birkhäuser Verlag, Basel, 1999.

[7] M. Linckelmann, *The orbit space of a saturated fusion system in contractible,* preprint.

[8] D. Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group,* Advances in Mathematics **28** (1978), 101–128.

[9] G. Robinson, *Amalgams, blocks, weights, fusion systems and finite simple groups,* Journal of Algebra **314** (2007), 912–923.

[10] E. H. Spanier, *Algebraic Topology,* McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[11] P., Symonds, *The orbit space of the p-subgroup complex is contractible,* Commentarii Mathematici Helvetici **73** (1998), 400–405.

[12] J. Thévenaz and P. J. Webb, *Homotopy equivalence of posets with a group action,* Journal of Combinatorial Theory. Series A **56** (1991), 173–181.

[13] T. tom Dieck, *Transformation Groups,* de Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter & Co., Berlin, 1987.